CSC 2421H : Graphs, Matrices, and Optimization Lecture 6 : October 22, 2018

Electrical Networks and their Applications to Random Walks

Lecturer: Sushant Sachdeva *Scribe: Calum MacRury*

1 Electrical Networks

Let us suppose that $G = (V, E)$ is an undirected graph, for which $|V| = n$ and $|E| = m$. It will be convenient to assume in this section that its edges are of the form $\{u, v\}$ for $u, v \in V$, highlighting the fact that they are undirected. We shall later replace each edge $e = \{u, v\} \in E$, with one of (u, v) or (v, u) to indicate the direction we wish to assign to e. In the former case, we say that u is directed to v, and in the latter case, we say that v is directed to u.

If we additionally specify a pair of distinct vertices $s, t \in V$, then we may consider an *electrical current* on G. Formally, this is a vector $\mathbf{f} \in \mathbb{R}^E$. Intuitively, the current assigned to an edge should also have a direction associated to it. In order to specify these directions, we first assign an arbitrary orientation to the edges of G. If $\{u, v\} \in E$, then we can orient (direct) it from u to v by replacing it with the directed edge (u, v), and by writing $u \to v$. We then adopt the following convention for determining the direction of the $|f_e|$ units of current for the (directed) edge $e = (u, v) \in \tilde{E}$:

- If $f_{u,v} > 0$, then we say that $|f_{u,v}|$ units of current move from u to v.
- If $f_{u,v} < 0$, then we that $|f_{u,v}|$ units of current move from v to u.
- If $f_{u,v} = 0$, then 0 units of current move in either direction.

Using this convention, the vector $\mathbf{f} \in \mathbb{R}^E$ is sufficient to describe the directions of the electrical current on G.

Now that we have formalized our notation for representing electrical currents, we consider a number of *conservation conditions* known as Kirchoff's Laws. Intuitively, these laws say that for each *internal node* (i.e $v \neq s, t$), the total amount of current entering v is equal to the total amount of current leaving it. That is, formally we have

$$\sum_{u: v \to u} f_{u,v} - \sum_{w: w \to v} f_{w,v} = 0,$$

for each $v \in V$, $v \neq s, t$.

Of course, if we consider s to be the *source* of the circuit, and t to be its *sink*, then we can generalize these constraints to include s and t as well. To do so, let us suppose that $\gamma \geq 0$ is the total external current entering the source s. In this case, we have that

$$\sum_{u: v \to u} f_{u,v} - \sum_{w: w \to v} f_{w,v} = \begin{cases} \gamma & \text{if } v = s \\ -\gamma & \text{if } v = t \\ 0 & \text{otherwise} \end{cases}$$

For our purposes, we shall normalize and assume that the amount of current entering s is 1; that is, $\gamma = 1$. In this case, there exists a unique matrix $\mathbf{B} \in \mathbb{R}^{n \times m}$ encoding the constraints we impose on $\mathbf{f}$. We have that $\mathbf{f}$ satisfies the conservation laws, if and only if

$$\mathbf{Bf} = \mathbb{1}_s - \mathbb{1}_t.$$

It is easy to check that for $(u, v) \in E$, if $\mathbf{b}_{u,v} := \mathbb{1}_u - \mathbb{1}_v$, then

$$\mathbf{B} = [\mathbf{b}_{e_1} \dots \mathbf{b}_{e_m}],$$

provided $e_1, \dots, e_m$ are the (directed) edges of G.

Let us now additionally associate a *resistance* r_e to each edge $e \in E$. If we consider the vector $\mathbf{r} := (r_e)_{e \in E}$, then we can define the diagonal matrix $\mathbf{R} \in \mathbb{R}^{m \times m}$, by setting $\mathbf{Diag}(\mathbf{R}) = \mathbf{r}$.

We can use $\mathbf{R}$ to define the *energy* of the current $\mathbf{f}$. This is defined to be,

$$\sum_{e \in E} \frac{1}{2} r_e f_e^2 = \frac{1}{2} \mathbf{f}^T \mathbf{R} \mathbf{f}.$$

With this definition, we can consider a natural optimization problem, in which $\mathbf{b}$ is taken to be a fixed vector within $\mathbb{R}^V$, and $\mathbf{B}$ is the matrix defined above:

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2} \mathbf{f}^T \mathbf{R} \mathbf{f} \\ \text{subject to} \quad & \mathbf{Bf} = \mathbf{b} \\ & \mathbf{f} \in \mathbb{R}^E \end{aligned} \tag{1}$$

We say that a current $\mathbf{f}$ is an *electrical current*, provided it is an optimum solution to OP 1.

Proposition 1.1. *If* $\mathbf{f}$ *is an optimum solution to OP 1 then* $\mathbf{Rf}$ *is orthogonal to* $\Delta \in \mathbb{R}^m$*, provided* $\mathbf{B}\Delta = 0$.

Proof. Assume $\mathbf{B}\Delta = 0$. Given $\epsilon \in \mathbb{R}$, we know that $\mathbf{f} + \epsilon\Delta$ is a feasible solution to OP 1, as

$$\mathbf{B}(\mathbf{f} + \epsilon\Delta) = \mathbf{Bf} + \epsilon\mathbf{B}\Delta = \mathbf{Bf} = b.$$

On the other hand, we know that $\mathbf{f}$ is an optimum solution so,

$$\frac{1}{2}(\mathbf{f} + \epsilon\Delta)^T \mathbf{R} (\mathbf{f} + \epsilon\Delta) \geq \frac{1}{2} \mathbf{f}^T \mathbf{R} \mathbf{f}.$$

After simplification, it follows that

$$\epsilon(\Delta^T \mathbf{Rf} + \epsilon \frac{1}{2} \Delta^T \mathbf{R} \Delta) \geq 0, \tag{2}$$

for all ϵ. Notice also that the entries of $\mathbf{R}$ are nonnegative, so $\Delta^T \mathbf{R} \Delta \geq 0$. We consider two cases: $\epsilon > 0$ and $\epsilon < 0$. First, consider $\epsilon > 0$. Then,

$$\Delta^T \mathbf{Rf} + \epsilon \frac{1}{2} \Delta^T \mathbf{R} \Delta \geq 0. \tag{3}$$

Now assume $\Delta^T \mathbf{Rf} < 0$. Then, let $\epsilon_0 = -\frac{2\Delta^T \mathbf{Rf}}{\Delta^T \mathbf{R}\Delta}$. Note that $\epsilon_0 > 0$ since $\Delta^T \mathbf{Rf} < 0$. This means that

$$\epsilon < \epsilon_0 \Rightarrow \Delta^T \mathbf{Rf} + \epsilon \frac{1}{2} \Delta^T \mathbf{R} \Delta < 0,$$

which is a contradiction. So, the assumption must be false and it must be true that $\Delta^T \mathbf{Rf} \geq 0$.

Now consider the case where $\epsilon < 0$. Then,

$$\Delta^T \mathbf{Rf} + \epsilon \frac{1}{2} \Delta^T \mathbf{R} \Delta \leq 0. \tag{4}$$

Now assume $\Delta^T \mathbf{Rf} > 0$. Then, let $\epsilon_0 = -\frac{2\Delta^T \mathbf{Rf}}{\Delta^T \mathbf{R}\Delta}$. Note that $\epsilon_0 < 0$ since $\Delta^T \mathbf{Rf} > 0$. This means that

$$\epsilon > \epsilon_0 \Rightarrow \Delta^T \mathbf{Rf} + \epsilon \frac{1}{2} \Delta^T \mathbf{R} \Delta > 0,$$

which is a contradiction. So, the assumption must be false and it must be true that $\Delta^T \mathbf{Rf} \leq 0$.

From the above cases, the only valid possibility is that $\Delta^T \mathbf{Rf} = 0$, proving that $\mathbf{Rf}$ is indeed orthogonal to Δ. □

Suppose that we now consider the subspace $C := \{\Delta \in \mathbb{R}^E : \mathbf{B}\Delta = 0\}$. Clearly, C is the kernel of the matrix $\mathbf{B}$ (which is typically denoted by $\ker(\mathbf{B})$). Moreover, as a corollary to the above proposition, we get the following result:

Corollary 1.2. *If* $\mathbf{f}$ *is an optimal solution to Equation 2, then* $\mathbf{Rf}$ *is orthogonal to the kernel of* $\mathbf{B}$*; that is,* $\mathbf{f} \in C^\perp$ *in the above notation.*

We shall now consider some properties of the subspace C. Before doing so, let us observe a proposition regarding the matrix $\mathbf{B}$.

Proposition 1.3. *We have that rank*$(\mathbf{B}) = n - 1$ *if and only if* G *is connected (where we consider* G *as an undirected graph).*

Proof. It is a standard result from linear algebra that both the row and column spaces of $\mathbf{B}$ have the same dimension. This integer is defined precisely as the value of $\text{rank}(\mathbf{B})$.

As a consequence of this result, we know that since $\mathbf{B}^T$ is the transpose of $\mathbf{B}$, each matrix must have the same rank. It is therefore sufficient to show that the result holds for the matrix $\mathbf{B}^T$.

Observe that for any vector $\mathbf{x} \in \mathbb{R}^V$, we have that

$$(\mathbf{B}^T)_e = x_u - x_v,$$

for each directed edge $e = (u, v)$ of G. In particular, this shows that $\mathbf{B}^T \mathbb{1}_V = 0$, and so the kernel of $\mathbf{B}^T$ has dimension greater or equal to 1.

One can additionally show that if G is connected, then provided $\mathbf{B}^T \mathbf{x} = 0$, we must have that $\mathbf{x} \in \text{span}\{\mathbb{1}_V\}$. In other words, the dimension of the kernel of $\mathbf{B}^T$ is exactly 1. In this case, the rank theorem for matrices implies that $\text{rank}(\mathbf{B}^T) = n - 1$.

Generalizing the above results, we can prove that if $k(G) \geq 1$ is the number of components of G, then we have that $\text{rank}(\mathbf{B}^T) = n - k(G)$ (here the components are formed considering G as an undirected graph). This completes both directions of the proposition.

□

Lemma 1.4. *If G is connected, then $dim(C) = m - (n-1)$.*

Proof. This lemma is an immediate corollary of the previous result, combined with the rank theorem for matrices.

□

Lemma 1.5. *If $Im(\mathbf{B}^T) := \{\mathbf{B}^T\mathbf{x} : \mathbf{x} \in \mathbb{R}^V\}$, namely the image of the matrix $\mathbf{B}^T$, then $Im(\mathbf{B}^T) = C^\perp = ker(B)^\perp$.*

Before we prove this lemma, we remark that this statement is a property of all matrices - it is not specific to $\mathbf{B}$.

Proof. Observe that $\text{Im}(\mathbf{B}^T)$ is spanned by the column vectors of $\mathbf{B}^T$. As a result, a vector $\mathbf{x} \in \mathbb{R}^E$ is in $\text{Im}(\mathbf{B})^\perp$ if and only if it is orthogonal to all the column vectors of $\mathbf{B}^T$. On the other hand, we know that $\mathbf{x}$ satisfies this property if and only if it is in the kernel of $\mathbf{B}$. We may therefore conclude that,

$$\text{Im}(\mathbf{B}^T)^\perp = C.$$

Taking the orthogonal complement of both sides of the above equation, we may conclude that

$$\text{Im}(\mathbf{B}^T) = C^\perp,$$

thus completing the proof.

□

As a result of the preceding lemmas, we may conclude the following theorem:

Theorem 1.6. *If $\mathbf{f}$ is an optimum solution to OP 1, then there exists some $\mathbf{x} \in \mathbb{R}^E$ such that $\mathbf{f} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{x}$.*

Proof. Observe that since $\mathbf{f}$ is an optimum solution, we know that $\mathbf{Rf}$ is orthogonal to C. On the other hand, we know that $\text{Im}(B^T) = C^\perp$, so there exists some $\mathbf{x} \in \mathbb{R}^V$ for which $\mathbf{Rf} = \mathbf{B}^T\mathbf{x}$. As the matrix $\mathbf{R}$ is invertible, the result thus holds.

□

We know that if $\mathbf{f}$ is an optimum solution to OP 1, then we have that

$$\mathbf{b} = \mathbf{Bf} = \mathbf{BR}^{-1}\mathbf{B}^T\mathbf{x},$$

as $\mathbf{Bf} = \mathbf{b}$ and $\mathbf{f} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{x}$. We can interpret the vector $\mathbf{x}$ as specifying a *voltage* of x_v units for each vertex $v \in V$. By applying $\mathbf{R}^{-1}\mathbf{B}^T$ to this vector $\mathbf{x}$, we can then recover the electrical current $\mathbf{f}$ on the edges G, given that the resistances are defined by $\mathbf{R}$.

In the next section, we shall characterize exactly what these voltage vectors look like. Before we discuss how this can be done, we first observe that the matrix $\mathbf{BR}^{-1}\mathbf{B}^T$ correponds to the Laplacian of a specific weighted graph we now describe.

Lemma 1.7. *The matrix $\mathbf{BR}^{-1}\mathbf{B}^T$ is the Laplacian of $G = (V, E, w)$, where $w_e := \frac{1}{r_e}$ for each $e \in E$.*

Proof. Observe that if $R = I$, the identity matrix, then

$$\begin{aligned}\mathbf{B}\mathbf{B}^T &= \sum_{e=(u,v)\in E} (\mathbb{1}_u - \mathbb{1}_v)(\mathbb{1}_u - \mathbb{1}_v)^T \\ &= \sum_{e=(u,v)\in E} L_{u,v},\end{aligned}$$

where $L_{u,v}$ is the Laplacian of the edge $e = (u, v) \in E$.

If the matrix R is not the identity, then a similar computation shows that

$$\begin{aligned}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T &= \sum_{e=(u,v)\in E} \frac{1}{r_e}(\mathbb{1}_u - \mathbb{1}_v)(\mathbb{1}_u - \mathbb{1}_v)^T \\ &= \sum_{e=(u,v)\in E} \frac{1}{r_e} L_{u,v},\end{aligned}$$

where the matrix $\frac{1}{r_e} L_{u,v}$ is the Laplacian of the edge $e = (u, v)$ with weight $\frac{1}{r_e}$.

□

1.1 Moore-Pensoose Pseudo-Inverse

Suppose that we are given a symmetric $n \times n$ matrix $\mathbf{A}$ together with an n-vector $\mathbf{b}$, and we wish to solve the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ for solve variable $\mathbf{x} \in \mathbb{R}^n$. In the case in which $\mathbf{A}$ is invertible, the vector $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ is the unique solution to this equation. When the matrix $\mathbf{A}$ is not invertible, then we can define the *Moore-Pensoose Pseudo-Inverse* of $\mathbf{A}$. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathbf{A}$, then assume that $\psi_1, \ldots, \psi_n$ are orthonormal eigenvectors of $\mathbf{A}$. Using the spectral decomposition of $\mathbf{A}$, we know that

$$\mathbf{A} = \sum_{i=1}^{n} \lambda_i \psi_i \psi_i^T.$$

We then define

$$\mathbf{A}^+ := \sum_{i=1:\lambda_i \neq 0}^{n} \frac{1}{\lambda_i} \psi_i \psi_i^T,$$

as the *pseudo-inverse* of $\mathbf{A}$.

If we consider the specific case when we are given an undirected graph $G = (V, E)$, then we can consider the *pseudo-inverse* $\mathbf{L}^+$ of its Laplacian $\mathbf{L}$. In this case, assuming that $\lambda_1 \leq \ldots \leq \lambda_n$ are the eigenvalues of $\mathbf{L}$, then we have that

$$\mathbf{L}(\mathbf{L}^+\mathbf{b}) = \sum_{i=1:\lambda_i > 0}^{n} \psi_i \psi_i^T \mathbf{b}, \tag{5}$$

where $\psi_1, \ldots, \psi_n$ are orthonormal eigenvectors of $\lambda_1, \ldots, \lambda_n$. Observe that if we define the matrix $\Pi := \sum_{i=1:\lambda_i>0}^{n} \psi_i \psi_i^T$, then $\Pi \in \mathbb{R}^{V \times V}$. Moreover, Π is an *orthogonal projection* onto the subspace spanned by $\beta := \{\psi_i : \lambda_i > 0 \text{ and } 1 \leq i \leq n\}$. That is, $\Pi^2 = \Pi$, $\Pi^T = \Pi$ and $\mathrm{Im}(\Pi) = \mathrm{span}(\beta)$.

For our purposes, we are particularly interested in the case when G is connected. If this is true, then we know that the kernel of $\mathbf{L}$ is spanned by $\mathbb{1}_V$, and so $\lambda_i > 0$ for $i = 2, \ldots, n$. If we also assume that $\mathbf{b}$ is orthogonal to $\mathbb{1}_V$, then $\mathbf{b} \in \mathrm{span}(\beta)$, and so $\Pi(\mathbf{b}) = \mathbf{b}$.

Under these assumptions, observe that Equation 5 simplifies to

$$\mathbf{L}(\mathbf{L}^+\mathbf{b}) = \mathbf{b}.$$

We may therefore conclude that $\mathbf{x} = \mathbf{L}^+\mathbf{b}$ is a solution to the equation "$\mathbf{L}\mathbf{x} = \mathbf{b}$".

Lemma 1.8. *If $G = (V, E)$ is a connected graph, and $\mathbf{b} \in \mathbb{R}^V$ is orthogonal to $\mathbb{1}_V$, then*

$$\{\mathbf{x} \in \mathbb{R}^V : \mathbf{L}\mathbf{x} = \mathbf{b}\} = \{\mathbf{L}^+\mathbf{b} + \alpha \mathbb{1}_V : \alpha \in \mathbb{R}\}.$$

Remark 1.9. *If we orient the edges of G and consider currents on the edges of G, then the condition on $\mathbf{b}$ has a natural interpretation: We can think of the assumption $\mathbf{b}^T\mathbb{1}_V = 0$ as enforcing the constraint that the net current into the circuit must be 0. For example, in the case of a single source-sink pair (s, t), the vector $\mathbf{b} := \mathbb{1}_s - \mathbb{1}_t$ has exactly one unit of current entering s and one unit of current leaving t. Clearly, the orthogonality condition is satisfied in this case.*

Proof. Observe that if $\alpha \in \mathbb{R}$, then we have that

$$\mathbf{L}(\mathbf{L}^+\mathbf{b} + \alpha \mathbb{1}_V) = \mathbf{b},$$

as $\mathbb{1}_V$ is in the kernel of $\mathbf{L}$. We may therefore conclude that

$$\{\mathbf{L}^+\mathbf{b} + \alpha \mathbb{1}_V : \alpha \in \mathbb{R}\} \subseteq \{\mathbf{x} \in \mathbb{R}^V : \mathbf{L}\mathbf{x} = \mathbf{b}\}.$$

To see the other inclusion, assume that $\mathbf{x} \in \mathbb{R}^V$ is such that $\mathbf{L}\mathbf{x} = \mathbf{b}$. In this case, we have that $\mathbf{L}\mathbf{x} = \mathbf{b} = \mathbf{L}(\mathbf{L}^+\mathbf{b})$. Thus,

$$\mathbf{L}(\mathbf{L}^+\mathbf{b} - \mathbf{x}) = 0,$$

and so $\mathbf{L}^+\mathbf{b} - \mathbf{x} \in \ker(\mathbf{L})$. But G was assumed to be connected, so the kernel of $\mathbf{L}$ is spanned by $\mathbb{1}_V$. It follows that there exists some $\alpha_0 \in \mathbb{R}$ such that $\mathbf{L}^+\mathbf{b} - \mathbf{x} = \alpha_0 \mathbb{1}_V$. Thus, $\mathbf{x} = \mathbf{L}^+\mathbf{b} - \alpha_0 \mathbb{1}_V$, and so $\mathbf{x} \in \{\mathbf{L}^+\mathbf{b} + \alpha \mathbb{1}_V : \alpha \in \mathbb{R}\}$. This implies the other direction of the inclusion, and so the statement holds.

□

We conclude this section by remarking that if we orient the graph G as in the previous section, and specify a resistance matrix $\mathbf{R}$ on its edges, then we can consider the special matrix defined by $\mathbf{L} = \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T$, where the matrix $\mathbf{B}$ is derived from the orientation on G. As we saw previously, $\mathbf{L}$ is in fact a Laplacian matrix. If we set $\mathbf{b} := \mathbb{1}_s - \mathbb{1}_t$ for some source-sink pair $(s, t) \in V$, then we can interpret this vector as passing 1 unit of current into s, and 1 unit out of t. In particular, we know that $\mathbb{1}^T\mathbf{b} = 0$. Thus, if we consider solutions to the equation "$\mathbf{L}\mathbf{x} = \mathbf{b}$", we know that

$$\{\mathbf{x} \in \mathbb{R}^V : \mathbf{L}\mathbf{x} = \mathbf{b}\} = \{\mathbf{L}^+\mathbf{b} + \alpha \mathbb{1}_V : \alpha \in \mathbb{R}\},$$

by Lemma 1.8. Observe that if $\mathbf{x} := \mathbf{L}^{+}\mathbf{b} + \alpha \mathbb{1}_V$ for some $\alpha \in \mathbb{R}$, then $\mathbf{x}$ specifies voltages on the vertices of G. Moreover, by applying $\mathbf{R}^{-1}\mathbf{B}^T$ to $\mathbf{x}$, we can recover the current

$$\mathbf{f} = \mathbf{R}^{-1}\mathbf{B}^T(\mathbf{L}^{+}\mathbf{b} + \alpha \mathbb{1}_V).$$

Of course, $\mathbf{B}^T \mathbb{1}_V = 0$, so this means that $\mathbf{f} = \mathbf{R}^{-1}\mathbf{B}^T(\mathbf{L}^{+}\mathbf{b})$. In particular, the current $\mathbf{f}$ must be an optimum solution to OP 1 by Theorem 1.6. In other words, any voltage solution of the above form induces an electrical current through G; that is, a current whose energy is minimum.

1.2 Random Walks

Let us consider a weighted undirected graph $G = (V, E, \mathbf{w})$, together with a pair of distinct nodes $s, t \in V$. If we assume that G is connected, and start a (non-lazy) random walk at node s, then we can define $h(s,t)$ to be the expected number of steps for the walk to reach t for the first time. One can show that this value is finite, no matter which nodes are chosen.

If we fix the node t, then we can define the vector $\mathbf{h}_t \in \mathbb{R}^V$, where

$$\mathbf{h}_t(s) := h(s,t),$$

for each $s \in V(G)$. We first observe that for $x \neq t$,

$$\mathbf{h}_t(x) = 1 + \sum_{y:\{x,y\} \in E} \frac{w_{x,y}}{\deg(x)} \mathbf{h}_t(y),$$

and $\mathbf{h}_t(t) = 0$. As a consequence, we know that for each $x \neq t$,

$$\deg(x)\mathbf{h}_t(x) = \deg(x) + \sum_{y:\{x,y\} \in E} w_{x,y}\mathbf{h}_t(y).$$

In vector notation, provided $x \neq t$, we can express this as

$$(\mathbf{D}\mathbf{h}_t)(x) = (\mathbf{D}\mathbb{1}_V)(x) + (\mathbf{A}\mathbf{h}_t)(x),$$

provided $\mathbf{A}$ is the adjacency matrix of G, and $\mathbf{D}$ is its degree matrix. Observe then that

$$(\mathbf{L}\mathbf{h}_t)(x) = (\mathbf{D}\mathbb{1}_V)(x),$$

for all $x \neq t$, where $\mathbf{L}$ is the Laplacian of G. On the other hand, we know that

$$0 = \mathbb{1}_V^T \mathbf{L}\mathbf{h}_t = \sum_{x \neq t} (\mathbf{L}\mathbf{h}_t)(x) + (\mathbf{L}\mathbf{h}_t)(t),$$

as $\mathbb{1}_V$ is in the kernel of $\mathbf{L}$, and is thus orthogonal to $\mathbf{L}\mathbf{h}_t$ (check this). It follows that

$$(\mathbf{L}\mathbf{h}_t)(t) = -\sum_{x \neq t} \deg(x) = \deg(t) - \mathbb{1}^T\mathbf{D}\mathbb{1}.$$

If we define $\mathbf{b} := \mathbf{D}\mathbb{1}_V - (\mathbb{1}^T\mathbf{D}\mathbb{1})\mathbb{1}_t$, then this implies that

$$\mathbf{L}\mathbf{h}_t = \mathbf{b}.$$

We may therefore use Lemma 1.8 to conclude that

$$\mathbf{h}_t = \mathbf{L}^+(\mathbf{D}\mathbb{1} - (\mathbb{1}^T\mathbf{D}\mathbb{1})\mathbb{1}_t) + \alpha\mathbb{1},$$

for some $\alpha \in \mathbb{R}$, as $\mathbf{b}$ is orthogonal to $\mathbb{1}$. Observing that $\mathbf{h}_t(t) = 0$, we may conclude that

$$\mathbb{1}_t^T\mathbf{h}_t = \mathbb{1}_t^T(\mathbf{L}^+\mathbf{b} + \alpha\mathbb{1}) = 0.$$

Thus, $\alpha = -\mathbb{1}_t^T\mathbf{L}^+\mathbf{b}$. If we fix some $s \in V$, then this implies that

$$\begin{aligned} h(s,t) =& \mathbb{1}_s^T\mathbf{h}_t \\ =& \mathbb{1}_s^T\mathbf{L}^+\mathbf{b} + \alpha\mathbb{1}_s^T\mathbb{1} \\ =& \mathbb{1}_s^T\mathbf{L}^+\mathbf{b} - \mathbb{1}_t^T\mathbf{L}^+\mathbf{b} \\ =& (\mathbb{1}_s - \mathbb{1}_t)^T\mathbf{L}^+(\mathbf{D}\mathbb{1} - (\mathbb{1}^T\mathbf{D}\mathbb{1})\mathbb{1}_t), \end{aligned}$$

giving us a convenient expression for the hitting time vector $\mathbf{h}_t$.

1.3 Commute Time

In addition to hitting times on undirected weighted graphs, we can also define commute times. If $G = (V, E, \mathbf{w})$ is connected, then provided $s, t \in V$, we can define the commute time from s to t as

$$C(s,t) := h(s,t) + h(t,s).$$

By our results from the previous section, we know that

$$h(s,t) = (\mathbb{1}_s - \mathbb{1}_t)^T\mathbf{L}^+(\mathbf{D}\mathbb{1} - (\mathbb{1}^T\mathbf{D}\mathbb{1})\mathbb{1}_t)$$

Thus, after simplification

$$C(s,t) = [(\mathbb{1}_s - \mathbb{1}_t)^T\mathbf{L}^+(\mathbb{1}_s - \mathbb{1}_t)](\mathbb{1}^T\mathbf{D}\mathbb{1}).$$

We remark that the vector $\mathbf{L}^+(\mathbb{1}_s - \mathbb{1}_t)$ can be interpreted as specifying voltages on the vertices of G. If 1 unit of current enters s and 1 unit leaves t, then an electrical current can be derived from this vector (see the end of the previous section for details).