Linear Programming Duality

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1. Introduction

This lecture dealt with the concept of duality in formulating linear optimizations. We began by analyzing a simple linear program, proving a lower bound by taking a linear combination of constraints. We then generalized this process and formulated the concepts of the dual and primal linear programs. We finished by stating and proving a theorem relating the solution of a dual to its primal known as Strong Duality.

2. A Toy Problem

Example 2.1 (A Simple Linear Program). Consider the following linear program:

$$\begin{aligned} \min \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \geq 4 \\ & 2x_1 + x_2 \geq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Is it possible to show a lower bound of 2.5 for the minimization problem using only arithmetic operations? What's the best we can do?

Solution. Notice that we can get the proposed bound of 2.5 by combining the last two constraints:

$$\begin{aligned} 2x_1 + x_2 &\geq 5 \\ x_2 &\geq 0 \\ 2x_1 + x_2 + x_2 &\geq 5 + 0 \\ 2(x_1 + x_2) &\geq 5 \\ x_1 + x_2 &\geq 2.5 \end{aligned}$$

But we can do even better. In fact, we can achieve the minimum bound by simply adding the first two constraints and dividing by three:

$$\begin{aligned} x_1 + 2x_2 &\geq 4 \\ 2x_1 + x_2 &\geq 5 \\ x_1 + 2x_2 + 2x_1 + x_2 &\geq 4 + 5 \\ 3(x_1 + x_2) &\geq 9 \\ x_1 + x_2 &\geq 3 \end{aligned}$$

This bound is achievable with $x_1 = 2, x_2 = 1$. The natural question is, then, is this process generalizable?

3. Duality of Linear Programs

So what exactly did we do? We took positive linear combinations of constraints. To formalize this, in general, any linear program LP can be written as:

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax \geq b \end{aligned}$$

where we have n variables and m constraints, and A is a real-valued m-by-n matrix. We can then pick some vector y of length m such that every entry is nonnegative, and consider

$$\sum_i y_i(a_i^\top x) \geq \sum_i y_i b_i \Rightarrow y^\top Ax \geq b^\top y$$

where the a_i are rows of the constraint matrix A.

Now suppose that we have $y \geq 0$ such that

$$A^\top y = c$$

Then we have

$$y^\top Ax = (A^\top y)^\top x = c^\top x \Rightarrow c^\top x \geq b^\top y$$

which gives us a lower bound on the minimization, as in the example. But this turns out to be another optimization problem, since we would like to find the maximum lower bound. This yields the follow concept of a *dual linear program*:

Definition 3.1 (Dual Linear Program). *Given a linear program LP with constraints $Ax \geq b$ and objective min $c^\top x$, we denote its* **Dual Linear Program (D)**

$$\begin{aligned} \max \quad & b^\top y \\ \text{s.t.} \quad & A^\top y = c \\ & y \geq 0 \end{aligned}$$

We call the original program the **Primal Linear Program (P)***.*

Lemma 3.2 (Weak Duality). *$\forall$ feasible x for (P) and feasible y for (D), we have $c^\top x \geq b^\top y$.*

Proof. Let $x^\star$ be the minimizer for (P), and $y^\star$ be the maximizer for (D). Then we have the following chain of inequalities:

$$c^\top x \geq c^\top x^\star \geq b^\top y^\star \geq b^\top y.$$

□

We adopt the following conventions:

unbounded minimization problems are assigned $-\infty$
unbounded maximization problems are assigned $+\infty$
infeasible minimization problems are assigned $+\infty$
infeasible maximization problems are assigned $-\infty$

Under this convention, the weak duality theorem always holds.

Fact 3.3. *The dual of a dual is equivalent to the primal.*

4. Strong Duality

Somewhat surprisingly, often we get *Strong Duality*, where the optimum values of the primal and the dual are equal. Consider the Primal (P) and Dual (D) programs as before. The following theorem states the result formally.

Theorem 4.1 (**Strong Duality**)**.** *If either (P) or (D) is feasible and bounded, then they are both feasible, bounded, and share the same optimum value.*

The full statement of the theorem, that takes into account all possible scenarios is the following:

Theorem 4.2 (**Strong Duality**)**.** *One of the following always holds:*

(1) Both (P) and (D) are infeasible.
(2) (P) is unbounded $\Rightarrow$ (D) is infeasible.
(3) (D) is unbounded $\Rightarrow$ (P) is infeasible.
(4) There exists an $x^\star$ feasible for (P) and a $y^\star$ feasible for (D) s.t. $c^\top x^\star = b^\top y^\star$.

For cases 1-3, the result follows by weak duality. So we assume cases 1-3 don't hold. Thus, at least one of (P) or (D) is feasible and bounded. Since the dual of a dual is equivalent to the primal, WLOG we assume (P) to be feasible and bounded. This means we only need to prove one direction of the fourth statement.

Let $x^\star$ be the minimizer for (P), and $I := \{i \in [m] | a_i^\top x^\star = b_i\}$ be the set of tight constraints at $x^\star$. Before attemping to prove the theorem, we first introduce the concept of *complementary slackness.*

Lemma 4.3 (**Complementary Slackness**)**.** *Suppose (P) is feasible and bounded. Let $x^\star$ be the optimizer for (P). Then, any y feasible for (D), satisfies $b^\top y = c^\top x^\star$ if and only if*

$$\forall i, y_i > 0 \Rightarrow a_i^\top x^\star = b_i.$$

Proof. For any feasible y, we define the **duality gap** to be $c^\top x^\star - b^\top y$. By weak duality, we know that

$$c^\top x^\star - b^\top y \geq 0.$$

Recall that $A^\top y = c$, so that we actually have

$$0 \leq y^\top A x^\star - y^\top b = \sum_i y_i (a_i^\top x^\star - b_i).$$

Since $Ax^\star \geq b$, and $y \geq 0$, each term in the above sum is positive. If $c^\top x^\star = b^\top y^\star$, then

$$\sum_i y_i (a_i^\top x^\star - b_i) = 0$$

implies that each term in the sum is exactly zero. Thus, we have

$$\forall i, y_i^\star > 0 \Rightarrow a_i^\top x^\star = b_i$$

as desired. □

This lemma shows that $\exists y^\star$ feasible s.t. $c^\top x^\star = b^\top y^\star \Leftrightarrow \exists y^\star$ feasible s.t. $i \notin I \Rightarrow y_i = 0$. So our theorem is equivalent to proving that there exists a feasible y satisfying complementary slackness.

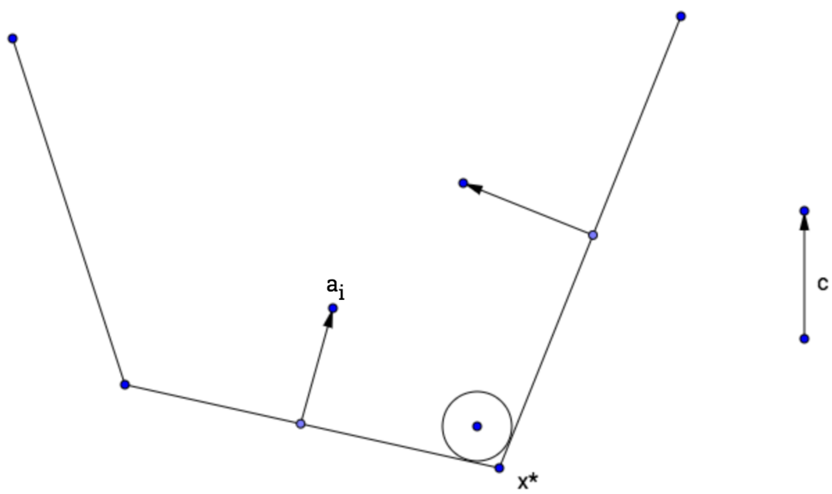

Figure 1. Situation described above.

Informal proof. Consider the following informal "proof". Consider the polytope of the given constraints. Let each constraint represent a *wall* of the polytope. Assume the optimization function is pointing up. The dropping a really small ball into the polytope should see the ball landing at exactly the minimization point.

However, the only walls applying force are the tight constraints. The force applied to the ball by the wall corresponding to constraint $a_i x \geq b_i$ is along the direction a_i. Thus, the total force due to the walls is just $\sum_{i \in I} a_i y_i$, for some $y_i \geq 0$. Since the ball is at rest, so the net force must be zero, i.e.

$$\sum_{i \in I} a_i y_i = c$$

Then we can just set $y_i = 0$ for $i \notin I$, and we obtain $c = \sum_i a_i y_i,\ y_i \geq 0$, with y satisfying complimentary slackness, and hence proving the theorem. Let us now formalize these intuitions.

Proof of Theorem 3.4. As mentioned before, parts 1-3 of the theorem follow directly from weak duality and the definitions, so we concern ourselves only with the fourth statement. Assume to the contrary. Then let $c \notin K := \{\sum_{i \in I} a_i y_i | y_i \geq 0\}$. Notice that K is a closed, convex cone.

Lemma 4.4 (Farkas' Lemma). *If $c \notin K$, $\exists d \in \Re$ s.t. $d^\top c < 0$ and $\forall z \in K,\ d^\top z \geq 0$,*

Proof. A sketch is given here. The idea behind the lemma is to pick d carefully. Take any point p s.t. $p \in K$ and $||p - c||$ is equal to the minimum of $||z - c||$ over all $z \in K$. Now for any $z \in K$, the angle $\angle cpz$ is obtuse, so we can show that $(z - p)^\top (c - p) \leq 0$. The final point is to show that such a p exists. (One approach is to use a theorem of Weierstrass's.) $\square$

This lemma allows us completes the proof. Consider $x = x^\star + \epsilon d$ for small $\epsilon > 0$. Then take $i \in I$, so that

$$a_i^\top x = a_i^\top x^\star + \epsilon a_i^\top d$$

Then each $a_i \in K \Rightarrow d^\top a_i \geq 0$, and in particular,

$$a_i^\top x^\star + \epsilon a_i^\top d \geq a_i^\top x^\star = b_i$$

Now consider $i \notin I$. We know that

$$a_i^\top x^\star > b_i$$

so there exists some $\epsilon > 0$ s.t.

$$a_i^\top x = a_i^\top x^\star + \epsilon a_i^\top d > b_i$$

(Note: we have implicitly taken take epsilon minimum over all i. As such, this proof is valid only for finite constraint sets.)

But this is a contradiction, because

$$d^\top c < 0 \Rightarrow c^\top x = c^\top x^\star + \epsilon c^\top d < c^\top x^\star$$

violating the minimality of $x^\star$, which completes the proof. $\square$