CPSC 665 : An Algorithmist's Toolkit Lecture 15 & 16 : 2 & 4 Mar 2015

Laplacian Systems and Randomized Kaczmarz

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1. Introduction to Laplacian Systems

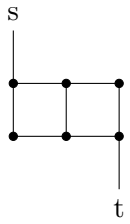
Let $G = (V, E)$ be a graph of m edges and v vertices. The matrix L is a Laplacian for graph G if $L = \sum_{(x,y)\in E}(e_x - e_y)^T(e_x - e_y)$ where e_x is the vector with a one in the x-th coordinate, and zeros elsewhere. We can notice a few useful properties:

(1) $L \succeq 0$
(2) $L\mathbf{1} = 0$, where $\mathbf{1}$ is the vector of all ones.
(3) If G is connected, L has only one zero eigenvalue.

Laplacian systems are of the form $Lv = b$. We can assume $b^T\mathbf{1} = 0$. Even though L is not invertible, if G is connected, $\exists! v$ such that $Lv = b$ and $v^T\mathbf{1} = 0$. Then $v = L^+b$, where L^+ is the pseudoinverse of L. In particular, if the spectral decomposition of $L = \sum_{i=1}^n \lambda_i u_i u_i^T$ has $\lambda_i > 0$ for $i \geq 2$, then the pseudoinverse $L^+ = \sum_{i=2}^n \frac{1}{\lambda_i} u_i u_i^T$.

2. Application to Electrical Systems

Laplacians can be used to solve electrical systems. Assume G represents a network of resisters, where each edge has resistance 1.



Say we send one unit of current along this network, with input at node s and output from node t. What is the flow?

We can pick an arbitrary orientation for each edge. The flow on each edge $f_{(x,y)}$ will be positive in the chosen direction and negative in the opposite direction. The net flow at a vertex x is the amount current flowing out minus the amount of current flowing in:

$$\sum_{y:(x,y)\in E} f_{(x,y)} - \sum_{y:(y,x)\in E} f_{(y,x)} = \begin{cases} 1 \text{ if } x = s \\ -1 \text{ if } x = t \\ 0 \text{ otherwise} \end{cases}$$

More generally, we can use an n by m incidence matrix B. Each column will represent an edge, (x, y) with a -1 the x-th position and a 1 in the y-th position. Then for any flow f, the net flow at vertex v is just the v-th component of Bf. In the case of unit flow from s to t, $Bf = e_s - e_t$. The system $Bf = b$ will be referred to as the flow constraints, where b gives the net flow out of each vertex.

The energy of a flow f is given by

$$\mathcal{E}(f) = \sum_{(x,y)\in E} \frac{1}{2} f_{(x,y)}^2 = \frac{1}{2}\|f\|^2$$

The electrical flow through a system will be the flow that minimizes energy while satisfying the flow constraints. So assuming the net flow out of each vertex is given by the vector b, we want to

pick $f^\star$ to minimize $\frac{1}{2}\|f\|^2$ such that $Bf = b$. We can solve this by minimizing the Lagrangian $L(f, v) = \frac{1}{2}\|f\|^2 - v^T[Bf - b]$. By the KKT conditions, $\frac{d}{df}L(f^\star, v^\star) = 0$, so $f^\star = B^T v^\star$. This factor $v^\star$ is simply the voltage of the system. Note that for each edge (x, y), $(B^T v)_{(x,y)} = v_x - v_y$, which is just Ohm's law. If $f^\star = B^T v^\star$ and $Bf^\star = b$, then $(BB^T)v^\star = b$. But BB^T is just the Laplacian L. So we can solve for the electrical flow by solving the Laplacian system $Lv^\star = b$.

Laplacian systems can be solved using the standard semidefinite programming techniques, such as gradient decent and conjugate gradient. If the graph G is an n cycle, then the condition number $\frac{\lambda_{\max}}{\lambda_{\min}}$ of the Laplacian is $\kappa(l) = \Omega(n^2)$ so conjugate gradient $\in \ O(n(m+n)\log(1/\epsilon))$. But we can do better than $O(n^2)$. By [Spielman and Teng, 2004], we can compute x such that $\left\|x - L^+ b\right\| \leq \epsilon \left\|L^+ b\right\|$ in $\tilde{O}(m\log(\frac{1}{\epsilon}))$, where $\tilde{O}(f(n)) = O(f(n)\text{poly}(\log(n))$. We will present a proof of this result from [Kelner et al., 2013], using the randomized Kaczmarz method.

3. Circulations

A vector $z \in \mathbb{R}^n$ such that $Bz = 0$ is called a circulation if $\forall x, (Bz)_x = 0$. In other words, the net flow at all vertices is 0.

As a simple example, consider a cycle c in the graph. Send 1 unit current along each edge in c- we will denote this flow by $\mathbf{1}_c$. Clearly, $\mathbf{1}_c^T f^* = 0$. This is just Kirchoff's Law. Other circulations are just linear combinations of cycles- that is $\{z|Bz = 0\} = \text{span}(\{\mathbf{1_c}|c \text{ is a cycle}\})$.

Let $T \subseteq G$ be a spanning tree for G. For any edge not in the spanning tree, adding that edge creates a unique cycle:

$$\forall e \in E(G) \setminus E(T), \exists! c_e \in T \cup \{e\}$$

This means an electric flow f must obey the following rules:

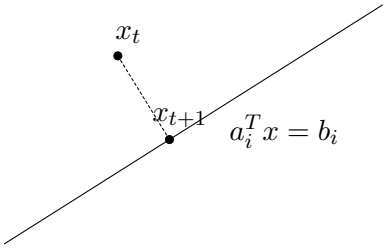
(1) Flow constraints: $Bf = b$
(2) Kirchoff's Law: $\forall e \in G \setminus T, \mathbf{1}_{c_e}^T f = 0$ where $\mathbf{1}_{c_e}$ is the unit circulation along the unique cycle in $T \cup \{e\}$.

4. Randomized Kaczmarz Method

The Kaczmarz method ['39] is a simple way of solving linear equations $Ax = b$. Start with arbitrary values for x. While there are constraints that are violated, pick a violated constraint $a_i^T x = b$ and set x to its projection along that constraint.

$$x_{k+1} = x_k + \frac{b_i - a_i^T x_k}{\|a_i\|^2} a_i$$

This is represented visually in the diagram below:



For the Kaczmarz method, indeed for any method of alternating projections, we know

$$\left\|x_{t+1} - x^*\right\|^2 \leq \left\|x_t - x^*\right\|^2 - \|x_{t+1} - x_t\|^2$$

where x^* lies at the intersection of all constraints. This property will be useful later on.

When solving electrical systems, the linear constraints along which we project will be the ones given by Kirchoff's law. This modified Kaczmarz method from [Strohmer and Vershynin, 2009] proceeds as follows:

(1) Start with f_0 such that $Bf_0 = b$, so that the flow constraints are obeyed.
(2) For $t = 1 \dots k$ sample $e \in G \setminus T$ with probability proportional to $|C_e|$ and set

$$f_t = f_{t-1} - \frac{\mathbf{1}_{C_e}^T f_{t-1}}{\|\mathbf{1}_{C_e}\|^2} \mathbf{1}_{C_e}$$

Since $B\mathbf{1}_{C_e} = 0$ the update step will never cause the flow constraints to be violated. More specifically, let $\tau = \sum_{e' \in G \setminus T} |C_{e'}|$ and the probability p_e of picking edge e be

$$\frac{|C_e|}{\sum_{e'} |C_{e'}|} = \frac{|C_e|}{\tau}$$

We can show that the energy of the flow goes down multiplicatively with each step in expectation. First, we show that any circulation is likely to have a large component in the basis direction $\mathbf{1}_{C_e}$ picked according to distribution p. In particular, let $\hat{C}_e := \frac{\mathbf{1}_{C_e}}{\|\mathbf{1}_{C_e}\|}$.

Lemma 4.1. *For any unit circulation $\vec{g}$*

$$\sum_{e \in E \setminus T} p_e (\vec{g}^T \hat{C}_e)^2 \geq \frac{\|\vec{g}\|^2}{\tau}$$

Proof. Let $P_{(a,b)}$ consist of the unique path in spanning tree T from a to b. Then $C_{(a,b)}$ is just $(a, b) \cup P_{(a,b)}$. We can deconstruct the equation using this property.

$$\sum_{e \in E \setminus T} p_e (\vec{g}^T \hat{C}_e)^2 = \frac{1}{\tau} \sum_{e \in E \setminus T} (g_e - \vec{g} P_e)^2$$

$$\geq \frac{1}{\tau} \left(\sum_{e \in E \setminus T} g_e^2 - 2 \sum_{e \in E \setminus T} g_e \cdot (\vec{g}^T P_e) \right) = \frac{1}{\tau} \left(\sum_{e \in E \setminus T} g_e^2 - 2\vec{g}^T \sum_{e \in E \setminus T} g_e P_e \right)$$

As $\vec{g}$ is a circulation, we know the values of its off tree edges from the values of its on tree edges.

$$\forall e' \in E : \left(- \sum_{ein E \setminus T} g_e P_e \right)_{e'} = \begin{cases} g_{e'} \text{ if } e' \in T \\ 0 \text{ otherwise} \end{cases}$$

Therefore:

$$\sum_{e \in E \setminus T} p_e (\vec{g}^T \hat{C}_e)^2 \geq \frac{1}{\tau} \left(\sum_{e \in E \setminus T} g_e^2 + 2 \sum_{e \in T} g_e^2 \right) \geq \frac{\|g\|^2}{\tau}$$

□

We can apply the lemma above to one iteration of the algorithm.

$$\mathbf{E}[(\hat{C}_e^T (f_t - f^*))^2] = \sum_{e \in E \setminus T} p_e (\hat{C}_e^T (f_t - f^*))^2 \geq \frac{1}{\tau} \cdot \left\| f_t - f^* \right\|^2$$

As this randomized method is simply a specialized version of the general Kaczmarz method, we know $\|x_{t+1} - x^*\|^2 \leq \|x_t - x^*\|^2 - \|x_{t+1} - x_t\|^2$. This means, for our algorithm:

$$\left\| f_t - f^* \right\|^2 - \left\| f_{t+1} - f^* \right\| \geq (\hat{C}_e^T f_t)^2$$

Therefore, the expected distance of the current solution to the optimal one decreases as

$$\mathbf{E}[\|f_t - f^*\|^2] \leq \left(1 - \frac{1}{\tau}\right)^t \|f_0 - f^*\|^2$$

5. Implementation

To implement the Kaczmarz algorithm we just need two operations:

(1) Query: computing $\mathbf{1}_{c_e}^T f_{t-1}$
(2) Update: computing $f_{t-1} - \alpha \mathbf{1}_{c_e}$.

We can modify these operations to efficiently store and update flows in T. Pick an arbitrary fixed vertex $s \in V$ to be the root. Adding α to every edge in the circulation $C_{(a,b)}$ is the same as adding α to (a, b), adding $-\alpha$ to every edge in the path from s to b, and adding α to every edge in the path from s to a.

Let $(d, T_0, T_1, \ldots, T_k)$ be a tree decomposition of T if the removal of $d \in V$ partitions T into subtrees $T_0 \ldots T_k$, where T_0 is rooted at s and contains d as a leaf, while the other T_i are rooted at d, and each T_i has at most $n/2 + 1$ vertices. For a spanning tree T rooted at s with $n \geq 2$ vertices, we can compute a tree decomposition in $O(n)$ time by starting at s and recursively picking the edge that leads to the largest subtree. Eventually we will find a d such that the size of all its subtrees have no more than $n/2$ vertices, making d the desired vertex separator. We know such a d exists thanks to Jordan in 1869.

Applying this idea recursively results in a separator decomposition tree of depth at most $\log(n)$. For each subtree rooted at d_i we can maintain two values: d_i-drop, the total potential drop on the path from s to d_i, and d_i-ext, the contribution to d_i drop from vertices beyond d. Flow along edges in $G \setminus T$ in are stored in a single array. This allows query and update operations to be performed in $O(\log(n))$ time.

6. Low Stretch Spanning Trees

[Kelner et al., 2013] showed that each iteration i of the algorithm computed a feasible $f_i \in \mathbb{R}^E$ such that

$$\mathbf{E}[\mathcal{E}(f_i)] - \mathcal{E}(f^\star) \leq \left(1 - \frac{1}{\tau}\right)^i \left(\mathcal{E}(f_0) - \mathcal{E}(f^\star)\right)$$

Therefore, for the Kaczmarz algorithm to complete quickly, we need spanning trees for which $\tau = \sum_{e' \in G \setminus T} |C_{e'}|$ is small. It turns out that τ is related to a property of spanning trees known as stretch. Remember that $P_{(a,b)}$ consists the unique path in spanning tree T from a to b. Then the stretch of some edge e is the length of P_e, and the stretch of T is the sum of the stretches of its edges. This means that $|C_e| = 1 + \text{stretch}(e)$, and consequently that $\tau = \text{stretch}(T) + m - 2n + 2$.

[Abraham and Neiman, 2012] gives an algorithm to construct low stretch spanning trees. Specifically, there exists a spanning tree $T \subseteq G$ such that

$$\sum_{e \in G \setminus T} \|C_e\| = O(m \log(n) \log(\log(n)))$$

which can be constructed in time $O(m \log(n) \log(\log(n)))$. If we use such a spanning tree with the algorithm from [Kelner et al., 2013], in $\tilde{O}(m \log(1/\epsilon))$ time the algorithm can compute f such that $\mathbf{E}[\mathcal{E}(f)] \leq (1 + \epsilon)\mathcal{E}(f^\star)$.

References

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